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On resonances for a harmonic oscillator coupled with massless bosons

C Billionnet†
Centre de Physique Théorique, Ecole Polytechnique, 91128 Palaiseau Cedex, France

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Abstract. We study a Hamiltonian coupling a harmonic oscillator to massless scalar bosons which may have arbitrary energy. For certain values of the parameters that this Hamiltonian contains, we prove that the poles of its resolvant matrix elements are not in one-to-one correspondence with the eigenstates of the isolated oscillator. This result raises the question of the validity of this correspondence, even for small coupling, in atom–radiation interaction.

1. Introduction

In this study, we give information on the analytic structure of the resolvant of the operator

$$H(\lambda, \mu, g) = a^*a \otimes 1 + 1 \otimes \mu H_{\text{rad}} + \lambda(a^* \otimes c(g) + a \otimes c^*(g)) \tag{1}$$

when $\lambda \in]0, 1]$ and μ is real and small. The Hamiltonian (1) acts in the tensor product of $\mathcal{H}_{\text{osc}}$, the space of states of a harmonic oscillator, with $\mathcal{H}_{\text{rad}}$, the space of states of a zero-mass boson field. g is a function such that $\lambda g(p)$ measures the strength of the coupling of the oscillator to the photon with momentum p. A particular assumption on g will be necessary, since this function will have to be continued analytically and we do not want it to change too much with this continuation. This assumption is given in the statement of proposition 3.1 and discussed before the proof. The shape of the graph of g, which must not be confused with the shape of the emission profile, will thus be supposed to be smooth.

This problem may appear as a rather academic question, due to the unphysical values of the parameters, and to the crude form of the Hamiltonian. However, since the model incorporates the photon field in a simple way, results in the ($\mu = 1$)-case should be easier to obtain than with the atom–radiation Hamiltonian and they could give information about the analytic structure of the resolvant in this latter problem. We think that our results might help in treating that physical $\mu = 1$ case.

The atom–radiation problem has of course been explored for a long time and, recently, important new results have been obtained [1–3]. In these papers, it is proved in a nonperturbative way that to an eigenvalue of the atomic Hamiltonian may be associated a complex value, a resonance, and that, in the neighbourhood of that eigenvalue, outside a cuspidal domain pointed at the resonance, analyticity properties of resolvant matrix elements are guaranteed. The study does not say anything about the analytic structure outside the neighbourhood or inside the cuspidal domain.

Here, in our model, we shall prove something that may appear surprising: one excited state of the oscillator gives rise to more than one pole of the resolvant matrix elements.

† E-mail address: billionnet@pth.polytechnique.fr

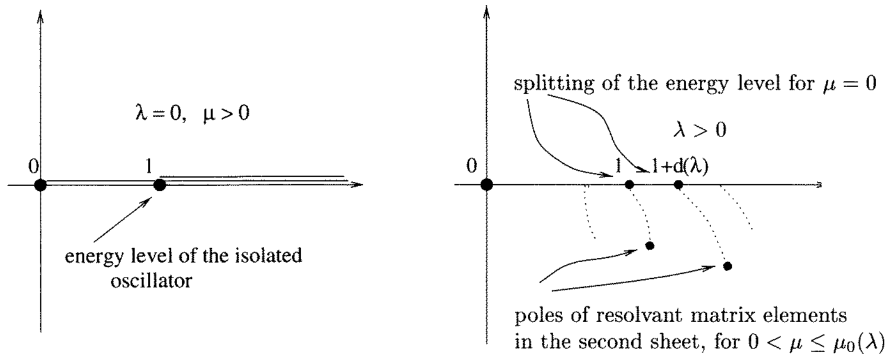

Figure 1.

The mechanism behind this result may be stated in the following way. As is the case in the atom–radiation problem, states of the system may contain any number of photons and the energy spectrum of the photon is $[0, \infty]$. These two facts have the consequence that energy states of the uncoupled oscillator–radiation system are infinitely degenerated since these states may consist of 1, 2, or n photons. The interaction may then be expected to separate the energies of such different states having the same unperturbed energy. In particular, if the energy of the radiation part is small, so that the energy of the state is close to an unperturbed energy of the oscillator (or of the atom), this would explain the presence of several poles corresponding to one unique oscillator level. We do not pretend that these different poles would remain close together when $\mu = 1$. The result we obtain for μ small is illustrated in figure 1; dotted curves are for the μ-dependence of the two poles.

This mechanism could operate in an atom–radiation interaction and has some analogy with AC Stark effect, the fact that the position of levels in an atom depends on the number of photons in the populated modes of the electromagnetic field. The $\mu = 1$ case in our model would be a simple system on which such a mechanism could be tested, mathematically.

To our knowledge, there is little work on the precise question we are interested in. A family of Hamiltonians to which (1) belongs has been thoroughly studied in [4]. Their spectrum may be determined according to the values of λ and μ. However, Arai does not treat the problem of the analytic structure of the resolvant matrix elements.

Compared with the atom–radiation Hamiltonian, $H(\lambda, \mu, g)$ has an important property in that it conserves n_{tot}, the sum of the excitation number of the harmonic oscillator and the boson number. This should make the $(\mu = 1)$-model simpler than nonrelativistic quantum electrodynamics (QED), while maintaining two important aspects of the physical problem which make it hard to grasp mathematically: the presence of an infinite number of particles and the fact that their mass is 0. From the purely mathematical point of view, it could also be interesting to study how the spectrum of $H(\lambda, 0, g)$ and that of $H(0, \mu, g)$ combine in the analytic structure of the resolvent of $H(\lambda, \mu, g)$.

Some partial information concerning poles of some matrix elements of the resolvant of (1) can be obtained from the fact that, reduced to the $(n_{\text{tot}} = 1)$-states, $H(\lambda, \mu, g)$ is the Friedrichs model. See, for instance, references on the subject in [5].

In a crude representation of atomic states, coupling of the atom with the quantum electromagnetic field transforms the real energies of the atomic levels into complex values, whose imaginary parts are the widths of the levels. In elementary textbooks, the displacements of the energy levels are often calculated with rules valid only for perturbations of isolated nondegenerated eigenvalues. We are not considering this case, as the eigenvalues lie inside the continuous spectrum, moreover at the edge of some parts of the continuum. The result of [3] makes the existence of complex values attributed to resonances rigorous.

It is generally admitted, at least when the coupling is small, that an eigenvalue of the unperturbed Hamiltonian does not give rise to an infinity of poles of the resolvent of the full Hamiltonian. This is the image that we question here. We will see that new features seem to appear when the number of bosons taken into account is greater than one. The precise formulation will be stated in the conclusion.

A refined description of atomic states has to take the strength of the coupling into account. It may modify the simple correspondence just mentioned between the real energies of the uncoupled atom (or oscillator) and complex energies of the coupled system. For instance, in [6], it can be seen in the Friedrichs model that when the coupling constant λ increases, some peaks appear in the probability amplitude, that were not present for small λ. In [4], it is shown that the spectrum of $H(\lambda, \mu, g)$ changes in nature when λ becomes greater than a certain critical value. So perturbation of the simple image may come from strong coupling effects although we believe that the perturbation which will appear in our study is of a different kind.

Let us note that one of the hypotheses made in [4] excludes the possibility of μ being too small. The Hamiltonian H is studied under that condition, which is alright for the physical problem, if one looks only at the spectrum of H. However, if information on the analytic structure of its resolvant is sought, then the ($\mu \simeq 0$)-behaviour may be relevant. It could be pertinent for the ($\mu = 1$)-problem, even if λ is small. This is the motivation behind this paper.

Another aspect has to be underlined. In the study of the resolvant in the complex plane, analyticity properties of g will have to be taken into account. This was already the case in the Friedrichs model (see for example [7]).

Let us now present in a more mathematical way the reason why one might be reluctant to accept the statement: 'for $\mu = 1$ and λ small, there is a one-to-one correspondence between the set of eigenvalues of a^*a and a set of complex numbers which would describe the excited states of the oscillator coupled to the bosons'. We will state here mathematical facts which sustain the motivation we expressed before in more physical terms.

We shall take the unusual point of view of considering (1) as a perturbation of $H(\lambda, 0, g)$ by the unbounded operator $1 \otimes \mu H_{\text{rad}}$. If $\mu = 0$, then the bosons have zero energy; therefore each eigenvalue $0, 1, 2, \ldots$ of $a^*a \otimes 1$ is infinitely degenerated as any number of bosons may be present, but the degeneracy is removed when the coupling term $\lambda(a^* \otimes c(g) + a \otimes c^*(g))$ is introduced. $H(\lambda, 0, g)$ is not lower bounded and its spectrum is given in the appendix; it no longer looks like that of a^*a. An excited state of the isolated oscillator, eigenvector of a^*a in $\mathcal{H}_{\text{osc}}$, gives rise, if λ is small, to an infinite number of eigenvectors of $H(\lambda, 0, g)$ in $\mathcal{H}_{\text{osc}} \otimes \mathcal{H}_{\text{rad}}$. One could then ask whether such energy-level splitting still occurs if $\mu \neq 0$. Our study is devoted to answering this question if μ is small. The result is stated in the proposition of section 5 and, it being understood that the eigenvalues are now changed into poles of matrix elements of the resolvant, the answer is yes. It raises the question: is that still true when $\mu = 1$?

As we can see, it is a multiboson effect, which is why we do not believe it is a strong coupling effect, although it seems to be the case when μ is small. However, in actual fact, there is no obvious connection between μ small and λ fixed in $[0, 1]$ and $\mu = 1$ and λ large in our problem.

Let us emphasize that this multiboson effect is already observed for the first excited level of the oscillator, although our Hamiltonian couples this level to 1-boson states only.

Indeed, taking the degeneracy of eigenvalue one into account entails considering more than one boson in the final state, because the state that we will shortly denote by $|1;1\rangle$ (the oscillator is in the first excited state and one boson is present) can evolve into $|0;2\rangle$ (the oscillator in the fundamental and two bosons present).

Many studies (for instance [8, 9]) on mathematically tractable models for atom–radiation interaction use 1-boson states only. This is made possible in two alternative ways. In the first one, Friedrichs model, (see [9] or [6, complement C_{III}]), the state-space of the system is not a tensor product clearly separating the matter and the radiation. One of the states in the Hilbert space of the system is viewed physically as the matter's excited state without any boson, but mathematically it is not the tensor product of a matter's state with the vacuum Ω_{rad} in some $\mathcal{H}_{\mathrm{rad}}$. It can evolve into a family of states indexed in a continuum, and these states are viewed as describing the oscillator (or the atom) in the fundamental, accompanied by one boson. However, these states are not a tensor product either. The Hilbert space does not contain any 2-boson states. In the second way, for instance in [9], the model does separate the matter and the radiation. In the state-space of the system, certain states are actually a tensor product of the excited state of the matter with 1-boson states, thus, seemingly, there should be 2-boson states, but the transition into a 2-boson state is forbidden by the Hamiltonian which is chosen deliberately so as to avoid considering an infinite number of bosons. In the problem we are treating here, 2-boson states play an essential role. This was also the case in an earlier study [10].

Clearly our Hamiltonian is defined on a space which contains an arbitrary number of bosons, and that feature usually makes the mathematical analysis difficult. The important point that will allow us to bypass that difficulty is the following. The fact that $H(\lambda, 0, g)$ has more than one eigenvalue in the neighbourhood of 1, for λ small, can be seen by only considering restrictions of $H(\lambda, 0, g)$ to two subspaces where the number of bosons is two at most. These subspaces are invariant by $H(\lambda, \mu, g)$. They are the eigenspaces associated with eigenvalues 1 and 2 of the operator $N_{\mathrm{tot}} = a^*a \otimes 1 + \textit{number}$ (bosons); they will be denoted by E_1 and E_2. So, turning now to $H(\lambda, \mu, g)$, as regards the poles of its resolvant matrix elements, corresponding to the preceding eigenvalues, we will still consider the restrictions of the operators to these subspaces, and, therefore, need not take an unbounded number of bosons into account.

Our problem contains another difficulty, which is underlined in [1]. It comes from the fact that the eigenvalues of $H(0, \mu, g)$ are not only embedded in the continuous spectrum, but are points of the boundary of parts of that continuum.

2. Notations and setting up

2.1. Notations

Let $\varphi^{(1)}$ be in $L^2(\mathbb{R})$ and $\varphi^{(2)}$ in $L^2_{\mathrm{sym}}(\mathbb{R}^2)$. We set $|i\rangle := (i!)^{-1/2}(a^*)^i\Omega_{\mathrm{osc}} \otimes \Omega_{\mathrm{rad}} |i;\varphi^{(1)}\rangle := (i!)^{-1/2}(a^*)^i\Omega_{\mathrm{osc}} \otimes \varphi^{(1)}$ and $|i;\varphi^{(2)}\rangle := (i!)^{-1/2}(a^*)^i\Omega_{\mathrm{osc}} \otimes \varphi^{(2)}$. Ω denotes the vacuum state.

E_1 is the subspace spanned by vectors of the form $|1\rangle$ or $|0;\varphi^{(1)}\rangle$ with $\varphi^{(1)} \in L^2(\mathbb{R})$. E_2 is the subspace spanned by vectors of the form $|2\rangle$, $|1;\varphi^{(1)}\rangle$ or $|0;\varphi^{(2)}\rangle$, $\varphi^{(1)} \in L^2(\mathbb{R})$, $\varphi^{(2)} \in L^2_{\mathrm{sym}}(\mathbb{R}^2)$.

Let $\mathcal{D}_n$ be the space of n-variable, symmetric, square integrable functions $\varphi^{(n)}$ such that, for $i = 1, \ldots n$, $p \to |p_i|\varphi^{(n)}(p_1, \ldots, p_n)$ is in $L^2(\mathbb{R}^n)$.

The domain D_1 of $H\upharpoonright_{E_1}$ is the space spanned by vectors of the form $|1\rangle$ and $|0;\varphi_1\rangle$ with $\varphi^{(1)} \in \mathcal{D}_1$. The domain D_2 of $H\upharpoonright_{E_2}$ is the space spanned by vectors of the form $|2\rangle$,

$|1; \varphi^{(1)}\rangle$ and $|0; \varphi^{(2)}\rangle$, $\varphi^{(1)} \in \mathcal{D}_1$, $\varphi^{(2)} \in \mathcal{D}_2$.

$H(\lambda, \mu, g) \restriction_{E_i}$ is self-adjoint on D_i for $i = 1, 2$.

We set, for $\Im z > 0$,

$$G_i(\lambda, \mu, z, g) := \langle i|[z - H(\lambda, \mu, g)]^{-1}|i\rangle = \langle i|[z - H(\lambda, \mu, g) \restriction_{E_i}]^{-1}|i\rangle. \tag{2}$$

It is customary to introduce operators R_i, called level-shift operators; they are defined, for $\Im z > 0$, by

$$R_i(\lambda, \mu, z, g) = \lambda V(g) \restriction_{E_i} + \lambda^2 V(g) Q_i [z - Q_i(H_0(\mu) + \lambda V(g)) \restriction_{E_i} Q_i]^{-1} Q_i V(g) \restriction_{E_i} \tag{3}$$

where

$$V(g) = a^* \otimes c(g) + a \otimes c^*(g) \qquad \text{and} \qquad H_0(\mu) = a^* a \otimes 1 + 1 \otimes \mu H_{\text{rad}}$$

(therefore $H(\lambda, \mu, g) = H_0(\mu) + \lambda V(g)$) and

$$Q_i = (1 - |i\rangle\langle i|) \restriction_{E_i} .$$

Setting

$$R_{i,i}(\lambda, \mu, z, g) = \langle i|R_i(\lambda, \mu, z, g)|i\rangle \tag{4}$$

we obtain (see [6])

$$G_i(\lambda, \mu, z, g) = [z - i - R_{i,i}(\lambda, \mu, z, g)]^{-1}. \tag{5}$$

We set $d(\lambda) = \frac{1}{2}(\sqrt{1 + 4\lambda^2} - 1)$. One has $d(\lambda) < \lambda^2$ if $\lambda \neq 0$ and $d(\lambda) \sim \lambda^2$, for small λ. d measures the oscillator's level splitting we mentioned in the introduction.

We also set $f_i(\lambda, \mu, z, g) = z - i - R_{i,i}(\lambda, \mu, z, g)$.

2.2. Setting up

Near $z = 1$, when $\lambda \in [0, 1]$ and μ is small, our aim is to derive that $G_2(\lambda, \mu, z, g)$ has a pole distinct from that of $G_1(\lambda, \mu, z, g)$ from the fact that $H(\lambda, 0, g)$ has two distinct eigenvalues in that region. Our tool will be Hurwitz' theorem (see [12]). Let us state it in the following form.

Let $f(\mu, z)$ be a function which, for all μ such that $0 \leqslant \mu < \mu_0$, is analytic in a disk $D(z_0, R)$, with centre z_0 and radius R, not depending on μ. Let us suppose that $\mu \to f(\mu, z)$ is continuous at 0, uniformly for $z \in D(z_0, R)$, and that $z \to f(0, z)$ does not vanish in D except at z_0, this zero being simple. Then there exists a function η, defined in $]0, R[$ and taking its values in $\mathbb{R}^+$, such that: $\forall \epsilon$ such that $0 < \epsilon < R$, $\forall \mu \in [0, \eta(\epsilon)[$, the function $z \to f(\mu, z)$ has a unique zero which is simple in the disk $D(z_0, \epsilon)$. Let us denote it by $z(\mu)$; moreover, the function $\mu \to z(\mu)$ is right-continuous at $\mu = 0$.

Poles of $G_i(\lambda, \mu, z, g)$ are zeros of $f_i(\lambda, \mu, z, g)$; they are known for $\mu = 0$ (see the appendix). Thus the function $f(\mu, z)$ in the theorem will be successively $f_1(\lambda, \mu, z, g)$ and $f_2(\lambda, \mu, z, g)$. We will thus study the analyticity of these two functions (sections 3.1 and 4.1) and their continuity at $\mu = 0$ (sections 3.2 and 4.2), in order to get the poles of $G_2(\lambda, \mu, z, g)$ (section 3.3) and $G_1(\lambda, \mu, z, g)$ (section 4.3).

3. Analyticity properties of $G_2(\lambda, \mu, z, g)$ near $z = 1$ and $\mu = 0$

Throughout this section, $H_2(\lambda, \mu, g)$ will denote the restriction of the operator (1) to E_2.

3.1. Analyticity of $R_{2,2}(\lambda, \mu, z, g)$ with respect to z

The entire section is devoted to proving the following.

Proposition 3.1. Let λ be fixed in]0,1] and g real valued on $]-\infty, +\infty[$ such that:

(1a) for some $\phi_0 \in]0, \pi/4]$, $p \to g(p)$ has analytic continuations g^+ in $\{p; p \neq 0, -\phi_0 \leqslant \arg p \leqslant \phi_0\}$ and g^- in $\{p; p \neq 0, \pi - \phi_0 \leqslant \arg p \leqslant \pi + \phi_0\}$,

(1b) $\forall f \in L^2(\mathbb{R})$, $\int g(\mathrm{e}^{-\theta} p) f(p)\,\mathrm{d}p$ and $\int [g(\mathrm{e}^{-\theta} p)]^2\,\mathrm{d}p$ exist and are analytic with respect to θ for $|\Im\theta| < \phi_0$,

(1c) $\exists M_g > 0$ such that $\forall\phi, -\phi_0 \leqslant \phi \leqslant \phi_0 : (\int_{-\infty}^{+\infty} p^2 |g(\mathrm{e}^{-\mathrm{i}\phi} p)|^2)^{1/2} < M_g$

(2) $\forall\theta |\Im\theta| \leqslant \phi_0$, $\eta_\theta(g) := 2^{-1/2}(\int_{-\infty}^{+\infty} |g(\mathrm{e}^{-\mathrm{i}\Im\theta} p)|^2\,\mathrm{d}p - 1)^{1/2} \leqslant C\lambda$ where $C < 4 \times 10^{-5} \sin\phi_0$.

Then there exists a neighbourhood $\mathcal{V}_\lambda$ of $z = 1$ and $\mu_1(\lambda) \in]0, 1]$ such that for all $\mu \in [0, \mu_1(\lambda)]$, $z \to R_{2,2}(\lambda, \mu, z, g)$ can be analytically continued in all $\mathcal{V}_\lambda$.

Before entering into the proof, let us make some comments, firstly on hypothesis 2, then on the method used.

Hypothesis 2 will be useful because analyticity properties of the resolvent matrix elements will be obtained by analytic continuation of g, the coupling function. We do not want the coupling λg to change too drastically with that analytic continuation. Thus, since it is of the order λ, we will ask the variation to be at most of a smaller order, i.e. λ^2; this is what hypothesis 2 says. Thus the order of the coupling is not changed. When g is analytically continued in the sector, its L_2 norm along the lines from the origin is supposed to vary slowly with the angle that the line forms with the real axis, but since the bound on the variation depends on λ, the class of admissible g functions depends on λ. Since λ may be chosen arbitrarily small, g may be forced to vary very slowly, in the L_2-norm sense, in the sectorial neighbourhood of the real axis, and we have to make sure that such classes of g are not void. Here is an example of a function which has the desired property in a sectorial neighbourhood of the positive axis: $g_n(z) = 2^{n/2}(n!)^{-1/2}\mathrm{e}^{-z^{1/n}}$. It can be shown that, for large n, with the L^2 norm on $[0, \infty]$, $(\|g_n(\mathrm{e}^{-\mathrm{i}\phi}.)\|^2 - 1)^{1/2}$ equals $\mathrm{O}(\phi n^{-1/2})$ and thus this quantity can be made smaller than $C\lambda$. The upper bound given on C is a crude one which could be improved. It has been calculated with λ arbitrary in $[0, 1]$ and it depends on the size of $\mathcal{V}_\lambda$ which is chosen.

Concerning the method of the proof, we are going to use a complex dilation e^θ, as is often the case in embedded eigenvalue perturbation problems. We have already said that our problem is of such a kind, since we could treat $\lambda(a^* \otimes c(g) + a \otimes c^*(g))$ as a perturbation of $H(0, \mu, g) = a^*a \otimes 1 + 1 \otimes \mu H_{\mathrm{rad}}$, although we are not going to proceed in that way. With that point of view, the unperturbed eigenvalue 1 is at the edge of one part of the continuous spectrum, namely the part corresponding heuristically to eigenvectors of the form $|1, p\rangle$, since p may be arbitrarily small. However, because of that particular location of the eigenvalue, rotating the spectrum of H_{rad} by changing p into $\mathrm{e}^\theta p$, and thus H_{rad} into $\mathrm{e}^{-\theta} H_{\mathrm{rad}}$, will not push the eigenvalue out of the continuous part of the spectrum of the dilated unperturbed Hamiltonian. The dilation will be useful in another way. Considering the functions $p \to g(\mathrm{e}^{-\theta} p)$ will reveal a simple means of taking advantage of the analyticity properties of g. In the simple case of $R_{1,1}(\lambda, \mu, z)$, which is presented in section 4, a well known method of proving analyticity properties of $G_1(\lambda, \mu, z)$ is to perform a contour deformation in an integral, and a way of doing it which can still be used for $G_2(\lambda, \mu, z)$ (whose expression is more complex) is to introduce the above functions $p \to g(\mathrm{e}^{-\theta} p)$.

Let us mention that this dilation, or analytic continuation, would not be necessary if g had a compact support. Indeed, restricting the operators to subspaces of E_1 and E_2

consisting of compactly supported functions would make $H_{\rm rad}$ bounded, which very much simplifies the demonstration. Theorem XII.11 of [5] may then be invoked. Calculations in that case are given in [11]. In this work, we let the energies of the bosons take any value.

The line of proof is as follows. As can be seen from (4) and (3), we are concerned with the existence and z-analyticity properties of the operator

$$L(\lambda,\mu,z,g) := [z - Q_2 H_2(\lambda,\mu,g) Q_2]^{-1} \tag{6}$$

leading to analyticity properties of its matrix element

$$R_{2,2}(\lambda,\mu,z,g) = 2\lambda^2\langle 1; g|[z - Q_2 H_2(\lambda,\mu,g)Q_2]^{-1}|1; g\rangle. \tag{7}$$

We will use the analyticity of g^+ and g^- in hypothesis 1 by introducing a θ-variable varying in $\mathcal{B} := \{\theta; |\Im\theta| \leqslant \phi_0\}$ and considering that, on $\mathbb{R}^+$ and $\mathbb{R}^-$, $g(p)$ is the value for $\theta = 0$ of $e^{-\theta/2} g^+(e^{-\theta} p)$ and $e^{-\theta/2} g^-(e^{-\theta} p)$, respectively. We will define $H_{2,\theta}(\lambda,\mu,g)$, $L(\lambda,\mu,z,\theta,g)$ and $R_{2,2}(\lambda,\mu,z,\theta,g)$, which coincide respectively with $H_2(\lambda,\mu,g)$, $L(\lambda,\mu,z,g)$ and $R_{2,2}(\lambda,\mu,z,g)$ when $\theta = 0$, and are analytic with respect to θ, for $\theta \in \mathcal{B}$, and z in certain domains. $L(\lambda,\mu,z,\theta,g)$ (and thus $R_{2,2}(\lambda,\mu,z,\theta,g)$), will first be defined for $\theta \in \mathcal{B}$ and z in a certain domain Λ included in the $\Im z > 0$ half-plane (section 3.1.2.1), then, after nailing down θ in $\mathcal{B}$, will be extended to the region $\Im z > 2^{3/2}\lambda\eta_\theta(g)$ (section 3.1.2.2), and, finally, to a neighbourhood of $z = 1$ (section 3.1.3). The important point is that, for $z \in \Lambda$, $R_{2,2}(\lambda,\mu,z,\theta,g)$ is constant with respect to θ; so it is in fact $R_{2,2}(\lambda,\mu,z,g)$ which has been continued through the real axis near $z = 1$, in our procedure. This is the desired result. This continuation through the real axis will be made possible by the fact that the antiself-adjoint part that $V(g)$ acquires when g is analytically continued can be controlled (see equation (28)). This is the purpose of hypothesis 2.

Let us now come to the proof in detail.

Proof. The dilation operates in the boson momentum space and its ratio is e^θ; θ is real for the moment but will soon be made complex. The unitary transformation induced in $\mathcal{H}_{\rm rad}$ by this dilation is denoted by A_θ and we set $\tilde{A}_\theta = 1 \otimes A_\theta$. On the 1-boson space, $A_\theta(f^{(1)})(p) = e^{-\frac{\theta}{2}} f^{(1)}(e^{-\theta}p)$. For real θ, we set

$$H_{2,\theta}(\lambda,\mu,g) := \tilde{A}_\theta H(\lambda,\mu,g)\tilde{A}_\theta^{-1} \restriction E_2 \tag{8}$$

defined on D_2 since D_2 is invariant by A_θ. As $A_\theta H_{\rm rad} A_\theta^{-1} = e^{-\theta} H_{\rm rad}$, one has

$$H_{2,\theta}(\lambda,\mu,g) = (a^* a \otimes 1 + \mu e^{-\theta} 1 \otimes H_{\rm rad} + \lambda V(g_\theta)) \restriction E_2 \tag{9}$$

where $g_\theta = A_\theta g$ and relations $A_\theta c(g) A_\theta^{-1} = c(g_\theta)$ and $A_\theta c^*(g) A_\theta^{-1} = c^*(g_\theta)$ have been used. The unitarity of $\tilde{A}_\theta$ yields, for $\Im z > 0$,

$$R_{2,2}(\lambda,\mu,z,g) = R_{2,2}(\lambda,\mu,z,\theta,g) \tag{10}$$

where

$$R_{2,2}(\lambda,\mu,z,\theta,g) := 2\lambda^2\langle 1; g_\theta|[z - Q_2 H_{2,\theta}(\lambda,\mu,g)Q_2]^{-1}|1; g_\theta\rangle \tag{10a}$$

a function which therefore does not depend on θ, if θ is in $\mathbb{R}$.

We set

$$L(\lambda,\mu,z,\theta,g) := [z - Q_2 H_{2,\theta}(\lambda,\mu,g)Q_2]^{-1}. \tag{11}$$

We are now going to make θ complex.

3.1.1. Analytic continuation of $H_{2,\theta}$ for $\theta \in \mathcal{B}$. Since we keep the Hilbert space unchanged, i.e. still E_2, A_θ does not make sense anymore if θ is complex; yet $H_{2,\theta}$, defined by (9) for θ real, may be analytically continued with respect to the θ-variable throughout the strip $-\phi_0 \leqslant \Im\theta \leqslant \phi_0$. With $g_\theta(p) = \mathrm{e}^{-\frac{\theta}{2}} g(\mathrm{e}^{-\theta} p)$, defined thanks to hypothesis 1 of proposition 3.1, this continuation is given by the following formula:

$$H_{2,\theta}(\lambda, \mu, g) = (a^* a \otimes 1 + \mu \mathrm{e}^{-\theta} 1 \otimes H_{\mathrm{rad}} + \lambda(a^* c(\overline{g_\theta}) + a c^*(g_\theta))) \restriction E_2. \tag{12}$$

As $H_{2,\theta}(\lambda, \mu, g)$ is unbounded, analyticity here is to be taken in the sense of Kato [13, section VII.2]. $H_{2,\theta}(\lambda, \mu, g)$ is defined on D_2 if $\mu \neq 0$ and on E_2 if $\mu = 0$. Neither $H_{2,\theta}(\lambda, \mu, g)$ nor even $H_{2,\theta}(\lambda, 0, g)$ are self-adjoint when θ is complex. Let us denote by V_2 the restriction of V to E_2, and set

$$V_{2,\theta}(g) = (a^* c(\overline{g_\theta}) + a c^*(g_\theta)) \restriction E_2 \tag{13a}$$

$$g_\theta^{(r)} = 2^{-1}(g_\theta + \overline{g_\theta}) \qquad g_\theta^{(i)} = (2i)^{-1}(g_\theta - \overline{g_\theta}) \tag{13b}$$

$$V_{2,\theta}^{(r)}(g) = V_2(g_\theta^{(r)}) \qquad V_{2,\theta}^{(i)}(g) = V_2(g_\theta^{(i)}). \tag{13c}$$

$V_{2,\theta}^{(r)}(g)$ and $V_{2,\theta}^{(i)}(g)$ are self-adjoint and $V_{2,\theta}(g) = V_{2,\theta}^{(r)}(g) + \mathrm{i} V_{2,\theta}^{(i)}(g)$; so

$$H_{2,\theta}(\lambda, \mu, g) = (a^* a \otimes 1 + \mu \mathrm{e}^{-\theta} 1 \otimes H_{\mathrm{rad}}) \restriction E_2 + \lambda(V_{2,\theta}^{(r)}(g) + \mathrm{i} V_{2,\theta}^{(i)}(g)). \tag{14}$$

$V_{2,\theta}(g)$ is bounded ($\|V_{2,\theta}(g)\| \leqslant 2^{3/2}\|g_\theta\|$) and $V_{2,\theta}(g)$ is bounded-holomorphic in θ for $\theta \in \mathcal{B}$. Therefore, $H_{2,\theta}(\lambda, \mu, g)$ is holomorphic for $\theta \in \mathcal{B}$ (see [13, VII.2, problem 1.2]). Thanks to hypothesis 2, the antiself-adjoint part of $V_{2,\theta}(g)$ can be controlled: $\|V_{2,\theta}^{(i)}(g)\| \leqslant 2^{3/2}\eta_\theta(g) < 2^{3/2} C\lambda$, because $\|g_\theta^{(i)}\| = \eta_\theta(g)$. (The equality $(g_\theta, \overline{g_\theta}) = 1$ has been used; this results from the analyticity of $(g_\theta, \overline{g_\theta})$, which has been assumed, and the fact that it has the value 1 for real θ. Also note that $\|g_\theta\|^2 > 1$).

3.1.2. Expression of $R_{2,2}(\lambda, \mu, z, g)$, for $\Im z > 2^{3/2}\lambda\eta_\theta(g)$, using the θ variable.

3.1.2.1. Expression of $R_{2,2}(\lambda, \mu, z, g)$ in $\Lambda \subset \{z; \Im z > 2^{3/2}\lambda\eta_\theta(g)\}$. Let us first remark that the important result in this section is not that $R_{2,2}(\lambda, \mu, z, g)$ is proved to be analytic for $\Im z > 2^{3/2}\lambda\eta_\theta(g)$, since it is known from the beginning to be analytic in $\Im z > 0$. The point is that $R_{2,2}(\lambda, \mu, z, g)$ will be written as an expression (see (15), with (10*a*) and (14)) which can be continued across the real axis, whereas that possibility was not obvious on the original one (4) (with (3)), and that the equality (15) is proved for $\Im z > 2^{3/2}\lambda\eta_\theta(g)$.

Let us determine a region Λ in the z-plane such that, $\forall z \in \Lambda$ and $\forall\theta \in \mathcal{B}$, formula (11), with $H_{2,\theta}$ now given by (14), defines an operator that we still denote by $L(\lambda, \mu, z, \theta, g)$. If $\Im(z\mathrm{e}^\theta) > 0$, $A := z - \mu\mathrm{e}^{-\theta}(1 \otimes H_{\mathrm{rad}})Q_2$ has a bounded inverse and besides, $B := Q_2 H_{2,\theta}(\lambda, 0, g) Q_2$ is bounded; it then suffices that $\|A^{-1}B\|$ should be smaller than 1 for the sum $A + B$ to be invertible. Since $\|A^{-1}\| \leqslant |\mathrm{e}^\theta||\Im(z\mathrm{e}^\theta)|^{-1}$, the relation $\|A^{-1}B\| < 1$ will be fulfilled as soon as $|\mathrm{e}^\theta||\Im(z\mathrm{e}^\theta)|^{-1}\|H_{2,\theta}(\lambda, 0, g)\| < 1$. Now, as a consequence of the hypotheses on g, there exists a, independent of λ, such that $\forall\theta$ such that $|\Im\theta| \leqslant \phi_0$, $\|H_{2,\theta}(\lambda, 0, g)\| < a$, if $\lambda < 1$. Let us set

$$\Lambda = \left\{z; 0 < \Re z < 2; \Im z > \frac{a + 2\sin\phi_0}{\cos\phi_0}\right\}.$$

It can be verified that $\forall z \in \Lambda, \forall\theta \in \mathcal{B}$, $\Im(z\mathrm{e}^\theta) > 0$ and $a|\mathrm{e}^\theta\|\Im(z\mathrm{e}^\theta)|^{-1} < 1$. Thus, for $z \in \Lambda$, (11) defines a θ-analytic family of bounded operators, $L(\lambda, \mu, z, \theta, g)$, when θ varies in $\mathcal{B}$.

As a consequence, the function $R_{2,2}(\lambda, \mu, z, ., g)$, originally defined by (10*a*) for $\Im z > 0$ and θ real, can be analytically continued throughout $\mathcal{B}$, for every z fixed in Λ.

Still for $z \in \Lambda$, $R_{2,2}(\lambda, \mu, z, \theta, g)$ is actually constant for θ real in $\mathcal{B}$; so it is constant in all $\mathcal{B}$ and

$$R_{2,2}(\lambda, \mu, z, g) = R_{2,2}(\lambda, \mu, z, \theta, g) \qquad \forall z \in \Lambda \qquad \forall \theta \in \mathcal{B}. \tag{15}$$

From now on, θ_0 being any fixed point in $\mathcal{B}$ satisfying $\Im\theta_0 = \phi_0$, we will put $\theta = \theta_0$. Let us show that $L(\lambda, \mu, z, \theta_0, g)$ exists $\forall z, \Im z > 2^{3/2}\lambda\eta_{\theta_0}(g)$.

3.1.2.2. Expression of $R_{2,2}(\lambda, \mu, z, g)$ in the whole half-plane $\Im z > 2^{3/2}\lambda\eta_{\theta_0}(g)$. Thanks to the positivity of H_{rad}, the bound for $\|V^{(i)}_{2,\theta_0}\|$ given in section 3.1.1 yields

$$\{\langle u|Q_2 H_{2,\theta_0}(\lambda, \mu, g)Q_2|u\rangle; u \in D(H_{2,\theta_0}), \|u\| = 1\} \subset \{z; \Im z < 2^{3/2}\lambda\eta_{\theta_0}(g)\}.$$

Then theorem V.3.2 of [13] tells us that if $\Im z > 2^{3/2}\lambda\eta_{\theta_0}(g)$,$[z - Q_2 H_{2,\theta_0}(\lambda, \mu, g)Q_2]$ is invertible since it is invertible for $z \in \Lambda$. Its inverse is analytic with respect to z for $\Im z > 2^{3/2}\lambda\eta_{\theta_0}(g)$ and it defines an analytic continuation of $z \to L(\lambda, \mu, z, \theta_0, g)$. Thus, by (15) and (10*a*), it provides an analytic continuation of $z \to R_{2,2}(\lambda, \mu, z, g)$ for $\Im z > 2^{3/2}\lambda\eta_{\theta_0}(g)$.

We are now going to continue the r.h.s. of (15) across the real z-axis.

3.1.3. Continuation of $R_{2,2}(\lambda, \mu, \ldots, g)$ into a neighbourhood of 1. First, let us sketch how this can be done, after setting, for $z = x + \mathrm{i}y$:

$$U_{\theta_0}(\lambda, z, g) := \mathrm{e}^{\theta_0}(z - Q_2(a^*a + \lambda V^{(r)}_{2,\theta_0}(g))Q_2) \tag{16}$$

$$W_{\theta_0}(\lambda, \mu, y, g) := \mathrm{e}^{\theta_0}(\mu\mathrm{e}^{-\theta_0} Q_2(1 \otimes H_{\mathrm{rad}}) + \mathrm{i}\lambda Q_2 V^{(i)}_{2,\theta_0}(g)Q_2 - \mathrm{i}y) \tag{17}$$

so that

$$z - Q_2 H_{2,\theta_0}(\lambda, \mu, g)Q_2 = -\mathrm{e}^{-\theta_0}(U_{\theta_0}(\lambda, x, g) - W_{\theta_0}(\lambda, \mu, y, g)). \tag{18}$$

(If $\mu \neq 0$, W_{θ_0} is unbounded, and its domain is D_2; it is E_2 if $\mu = 0$.)

We want an inverse $L'(\lambda, \mu, z, \theta_0, g)$ for $z - Q_2 H_{2,\theta_0}(\lambda, \mu, g)Q_2$, the l.h.s. of (18). If $U_{\theta_0}(\lambda, x, g)$ is invertible, its inverse being denoted by $T_{\theta_0}(\lambda, z, g)$, then, formally,

$$L'(\lambda, \mu, z, \theta_0, g) = \mathrm{e}^{\theta_0}[\mathrm{e}^{\theta_0} - T_{\theta_0}(\lambda, x, g)W_{\theta_0}(\lambda, \mu, y, g)]^{-1} T_{\theta_0}(\lambda, x, g) \tag{19}$$

and to prove the existence of $L'_{\theta_0}(\lambda, \mu, z, g)$, it is sufficient to show that the imaginary part of $T_{\theta_0}(\lambda, x, g)W_{\theta_0}(\lambda, \mu, y, g)$ is smaller than that of e^{θ_0}. The crucial point will be formula (29) which shows that the imaginary part is a bounded operator (whereas the real part is not). That the bound is small enough appears in formula (28) as a consequence of hypothesis 2. Then, once we have defined $L'(\lambda, \mu, z, \theta_0, g)$ for z in $\mathcal{V}_{1,\lambda}$, a neighbourhood of 1, we will check (section 3.1.3.3) that it is an analytic continuation in z of $L(\lambda, \mu, z, \theta_0, g)$, previously defined in $\Im z > 2^{3/2}\lambda\eta_{\theta_0}(g)$. Thus we will get an analytic continuation of $R_{2,2}(\lambda, \mu, ., g)$, near 1, across the real axis.

The following section is devoted to proving that $U_{\theta_0}(\lambda, x, g)$ is invertible in a neighbourhood of $x = 1$, and to obtaining a bound for the norm of its inverse (formula (23*b*)).

3.1.3.1. Definition and study of $T_{\theta_0}(\lambda, z, g)$. Let us set $q(\lambda, z, g_{\theta_0}^{(r)}) = z(z-1) - 2\lambda^2\|g_{\theta_0}^{(r)}\|^2$ and $q_1(\lambda, z, g_{\theta_0}^{(r)}) = z(z-1) - \lambda^2\|g_{\theta_0}^{(r)}\|^2$. It can be verified that the following formulae define the inverse of $U_{\theta_0}(\lambda, x, g)$ wherever the functions q^{-1}, q_1^{-1} and z^{-1} are defined:

$$T_{\theta_0}(\lambda, z, g)|2\rangle = z^{-1}|2\rangle \tag{20a}$$

$$\begin{aligned} T_{\theta_0}(\lambda, z, g)|1; \varphi\rangle &= q_1^{-1}(\lambda, z, g_{\theta_0}^{(r)})(z|1; \varphi\rangle + \sqrt{2}\lambda|0; g_{\theta_0}^{(r)} \vee \varphi\rangle \\ &\quad + \lambda^2(g_{\theta_0}^{(r)}, \varphi)q^{-1}(\lambda, z, g_{\theta_0}^{(r)})[z|1; g_{\theta_0}^{(r)}\rangle + \sqrt{2}\lambda|0; g_{\theta_0}^{(r)} \vee g_{\theta_0}^{(r)}\rangle]) \end{aligned} \tag{20b}$$

$$\begin{aligned} T_{\theta_0}(\lambda, z, g)|0; \varphi \vee \psi\rangle &= z^{-1}|0; \varphi \vee \psi\rangle + \frac{\lambda}{\sqrt{2}} z^{-1} q_1^{-1}(\lambda, z, g_{\theta_0}^{(r)}) \\ &\quad \times (z|1; (g_{\theta_0}^{(r)}, \varphi)\psi + (g_{\theta_0}^{(r)}, \psi)\varphi\rangle + \sqrt{2}\lambda|0; g_{\theta_0}^{(r)} \vee ((g_{\theta_0}^{(r)}, \varphi)\psi + (g_{\theta_0}^{(r)}, \psi)\varphi)\rangle \\ &\quad + 2\lambda^2(g_{\theta_0}^{(r)}, \varphi)(g_{\theta_0}^{(r)}, \psi)q^{-1}(\lambda, z, g_{\theta_0}^{(r)})(z|1, g_{\theta_0}^{(r)}\rangle + \sqrt{2}\lambda|0; g_{\theta_0}^{(r)} \vee g_{\theta_0}^{(r)}\rangle)). \end{aligned} \tag{20c}$$

In these formulae, our convention is $f \vee g = 2^{-1}(f \otimes g + g \otimes f)$.

We set $x_\pm(\lambda, g_{\theta_0}^{(r)}) := \frac{1}{2}(1 \pm \sqrt{1 + 8\lambda^2\|g_{\theta_0}^{(r)}\|^2})$ and $x_{1,\pm}(\lambda, g_{\theta_0}^{(r)}) := \frac{1}{2}(1 \pm \sqrt{1 + 4\lambda^2\|g_{\theta_0}^{(r)}\|^2})$; they are the zeros of $q(\lambda, x, g_{\theta_0}^{(r)})$ and $q_1(\lambda, x, g_{\theta_0}^{(r)})$ respectively. Thus $T_{\theta_0}(\lambda, z, g)$ is defined for $z \neq 0, x_\pm, x_{1,\pm}$. Let $\mathcal{V}_{1,\lambda}$ be the disk with centre 1 and radius $\lambda^2/2$. $\mathcal{V}_{1,\lambda}$ does not contain these points since $\|g_{\theta_0}^{(r)}\| = (1 + \eta_{\theta_0}(g)^2)^{1/2} > 1$. $U_{\theta_0}(\lambda, x, g)$ is thus defined in $\mathcal{V}_{1,\lambda}$.

An alternative expression for $T_{\theta_0}(\lambda, z, g)$ will be useful: in $\mathcal{L}(E_2)$, for $\Im z > 0$, let us set $G_0(z) = (z - Q_2(a^*a \otimes 1))^{-1}$ and

$$\begin{aligned} V_2^+ &:= G_0(z)Q_2(a \otimes c(g_{\theta_0}^{(r)})^*)Q_2G_0(z)Q_2(a^* \otimes c(g_{\theta_0}^{(r)}))Q_2 \\ V_2^- &:= G_0(z)Q_2(a^* \otimes c(g_{\theta_0}^{(r)}))Q_2G_0(z)Q_2(a \otimes c(g_{\theta_0}^{(r)})^*)Q_2. \end{aligned}$$

Then, for $\Im z > 0$, one may write $T_{\theta_0}(\lambda, z, g)$ in the form

$$\begin{aligned} T_{\theta_0}(\lambda, z, g) &= -G_0(z) + [1 - \lambda^2 V_2^+]^{-1}(1 + \lambda G_0(z)Q_2(a \otimes c(g_{\theta_0}^{(r)})^*)Q_2)G_0(z) \\ &\quad + [1 - \lambda^2 V_2^-]^{-1}(1 + \lambda G_0(z)Q_2(a^* \otimes c(g_{\theta_0}^{(r)}))Q_2)G_0(z). \end{aligned} \tag{21}$$

This formula may then be used to turn $T_{\theta_0}(\lambda, z, g)$ into the following operatorial form which, as formulae (20), does not have a pole at $z = 1$, unlike $G_0(z)$:

$$\begin{aligned} T_{\theta_0}(\lambda, z, g) &= q_1^{-1}\lambda(Q_2(a^* \otimes c(g_{\theta_0}^{(r)})) + (a \otimes c(g_{\theta_0}^{(r)})^*)Q_2) \\ &\quad + (zq_1)^{-1}\lambda^2 Q_2(1 - a^*a) \otimes c(g_{\theta_0}^{(r)})^* c(g_{\theta_0}^{(r)}) + q_1^{-1} z Q_2(a^*a \otimes 1) \\ &\quad + z^{-1}(1 - a^*a)^2 \otimes 1 + (qq_1)^{-1}\lambda^2(z|1; g_{\theta_0}^{(r)}\rangle\langle 1; g_{\theta_0}^{(r)}| \\ &\quad + \sqrt{2}\lambda|1; g_{\theta_0}^{(r)}\rangle\langle 0; g_{\theta_0}^{(r)} \vee g_{\theta_0}^{(r)}| + \sqrt{2}\lambda|0 g_{\theta_0}^{(r)} \vee g_{\theta_0}^{(r)}\rangle\langle 1; g_{\theta_0}^{(r)}| \\ &\quad + 2z^{-1}\lambda^2|0; g_{\theta_0}^{(r)} \vee g_{\theta_0}^{(r)}\rangle\langle 0; g_{\theta_0}^{(r)} \vee g_{\theta_0}^{(r)}|). \end{aligned} \tag{22}$$

(22) may also be verified directly from (20); we recall that the operators are restricted to E_2.

In order to show that $T_{\theta_0}(\lambda, z, g) \in \mathcal{B}(E_2)$, let us use (22). From $\|g_{\theta_0}^{(r)}\| > 1$, it follows that for $0 < \lambda < 1$ and $|z - 1| < \lambda^2/2$, one has $|z^{-1}| < 2$, $|q_1(\lambda, z, g_{\theta_0}^{(r)})| > \frac{\sqrt{5}-2}{4}\lambda^2$ and $|q(\lambda, z, g_{\theta_0}^{(r)})| > \lambda^2/4$. As a consequence, there exists C_1 such that

$$\|T_{\theta_0}(\lambda, z, g)\| \leqslant C_1\lambda^{-2}\|g_{\theta_0}^{(r)}\|^4 = C_1\lambda^{-2}(1 + \eta_{\theta_0}(g)^2)^2. \tag{23a}$$

One can verify that this equality is true with $C_1 = 300$. Since, by hypothesis, $\eta_{\theta_0}(g) < C$, we obtain

$$\|T_{\theta_0}(\lambda, z, g)\| \leqslant C_1(1 + C^2)^2\lambda^{-2} \qquad \forall z \in \mathcal{V}_{1,\lambda}. \tag{23b}$$

Thus T_{θ_0} is bounded-analytic for z in $\mathcal{V}_{1,\lambda}$.

3.1.3.2. Existence of $[\mathrm{e}^{\theta_0} - T_{\theta_0}(\lambda, x, g)W_{\theta_0}(\lambda, \mu, y, g)]^{-1}$. We decompose W_{θ_0} into its self-adjoint and antiself-adjoint parts:

$$\begin{aligned} W_{\theta_0} &= W_{\theta_0}^{(r)} + \mathrm{i}W_{\theta_0}^{(i)} \\ W_{\theta_0}^{(r)} &= \mu Q_2(1 \otimes H_{\mathrm{rad}}) - \lambda\Im(\mathrm{e}^{\theta_0})Q_2V_{2,\theta_0}^{(i)}Q_2 + y\Im(\mathrm{e}^{\theta_0}) \\ W_{\theta_0}^{(i)} &= \lambda\Re(\mathrm{e}^{\theta_0})Q_2V_{2,\theta_0}^{(i)}Q_2 - y\Re(\mathrm{e}^{\theta_0}). \end{aligned}$$

As $T_{\theta_0}(\lambda, x, g)$, defined for $|x-1| < \lambda^2/2$, is self-adjoint on E_2, then, in the decomposition

$$T_{\theta_0}(\lambda, x, g)W_{\theta_0}(\lambda, \mu, y, g) = A_{\theta_0}(\lambda, \mu, x, y, g) + \mathrm{i}C_{\theta_0}(\lambda, \mu, x, y, g) \tag{24a}$$

where $A_{\theta_0}(\lambda, \mu, x, y, g)$ and $C_{\theta_0}(\lambda, \mu, x, y, g)$ are self-adjoint, the antiself-adjoint part of $T_{\theta_0}(\lambda, x, g)W_{\theta_0}(\lambda, \mu, y, g)$ is

$$\begin{aligned} C_{\theta_0}(\lambda, \mu, x.y, g) = (2i)^{-1}&[T_{\theta_0}(\lambda, x, g), W_{\theta_0}^{(r)}(\lambda, \mu, y, g)] \\ &+2^{-1}[T_{\theta_0}(\lambda, x, g), W_{\theta_0}^{(i)}(\lambda, y, g)]_+ \end{aligned} \tag{24b}$$

where $[,]_+$ is the anti-commutator.

Let us show that there exists $b_g(\lambda, \mu, y, \theta_0) > 0$ such that

$$\forall x|x-1| < \lambda^2/2 \qquad \|C_{\theta_0}(\lambda, \mu, x, y, g)\| < b_g(\lambda, \mu, y, \theta_0). \tag{25}$$

By theorem IV.3.17 and formula V.(3.16) of [13], this will imply that the spectrum $\sigma(T_{\theta_0}(\lambda, x, g)W_{\theta_0}(\lambda, \mu, y, g))$ of $T_{\theta_0}(\lambda, x, g)W_{\theta_0}(\lambda, \mu, y, g))$ is contained in the strip of width b_g centred along the real axis, for such values of z.

Let us use (20). $\|g_{\theta_0}^{(r)}\|^{-1}a^*c(g_{\theta_0}^{(r)})$, $\|g_{\theta_0}^{(r)}\|^{-1}ac(g_{\theta_0}^{(r)})^*$, $\|g_{\theta_0}^{(r)}\|^{-2}c(g_{\theta_0}^{(r)})^*c(g_{\theta_0}^{(r)})$, $\|g_{\theta_0}^{(r)}\|^{-2}|1; g_{\theta_0}^{(r)}\rangle\langle 1; g_{\theta_0}^{(r)}|$, $\|g_{\theta_0}^{(r)}\|^{-3}|1; g_{\theta_0}^{(r)}\rangle\langle 0; g_{\theta_0}^{(r)} \vee g_{\theta_0}^{(r)}|$, and finally $\|g_{\theta_0}^{(r)}\|^{-4}|0; g_{\theta_0}^{(r)} \vee g_{\theta_0}^{(r)}\rangle\langle 0; g_{\theta_0}^{(r)} \vee g_{\theta_0}^{(r)}|$, as operators acting from D_2 into D_2, have a commutator with $1 \otimes H_{\mathrm{rad}}$ which can be extended to all E_2; these extensions are bounded in norm by $C_2'\|g_{\theta_0}^{(r)}\|^{-1}\|H_{\mathrm{rad}}g_{\theta_0}^{(r)}\|$, where C_2' is a number. It follows that there exists $C_2 \in \mathbb{R}^+$ such that $[T_{\theta_0}(\lambda, x, g), 1 \otimes H_{\mathrm{rad}}]$ is bounded by $C_2\lambda^{-2}\|g_{\theta_0}^{(r)}\|^3 \cdot \|H_{\mathrm{rad}}g_{\theta_0}^{(r)}\|$, uniformly for z such that $|x-1| < \lambda^2/2$. Now, since $T_{\theta_0}(\lambda, x, g)$ is uniformly bounded for these values of x and $V_{2,\theta_0}^{(i)}$ is bounded, then $[T_{\theta_0}(\lambda, x, g), W_{\theta_0}^{(r)}(\lambda, \mu, y, g)]$ is bounded, as well as $[T_{\theta_0}(\lambda, x, g), W_{\theta_0}^{(i)}(\lambda, \mu, y, g)]_+$; this proves that $C_{\theta_0}(\lambda, x, y, g)$ is bounded. To be more accurate, if $|x-1| < \lambda^2/2$,

$$\begin{aligned} \|[T_{\theta_0}(\lambda, x, g, W_{\theta_0}^{(r)}(\lambda, \mu, y, g)]\| &< \mu\lambda^{-2}C_2\|g_{\theta_0}^{(r)}\|^3 \cdot \|H_{\mathrm{rad}}g_{\theta_0}^{(r)}\| \\ &\quad +2^{\frac{5}{2}}\lambda\Im(\mathrm{e}^{\theta_0})\eta_{\theta_0}(g)\|T_{\theta_0}(\lambda, x, g\| \end{aligned} \tag{26a}$$

$$\|[T_{\theta_0}(\lambda, x, g, W_{\theta_0}^{(i)}(\lambda, \mu, y, g)]_+\| < 2(2^{3/2}\lambda\eta_{\theta_0}(g) + |y|)|\Re(\mathrm{e}^{\theta_0})|\|T_{\theta_0}(\lambda, x, g\|. \tag{26b}$$

This yields (25) with

$$b_g(\lambda, \mu, y, \theta_0) = 2^{-1}C_2\lambda^{-2}\mu\|g_{\theta_0}^{(r)}\|^3\|H_{\mathrm{rad}}g_{\theta_0}^{(r)}\| + \mathrm{e}^{\Re\theta_0}(2^{5/2}\lambda\eta_{\theta_0}(g) + |y|)\|T_{\theta_0}\|. \tag{27}$$

Now (25) and (27), together with the control on the antiself-adjoint part of $V(g)$, will enable us to get an upper bound for $b_g(\lambda, \mu, y, \theta_0)$, and thus to determine $\mathcal{V}_\lambda$, a neighbourhood of $z = 1$, and $\mu_1(\lambda) > 0$ such that

$$\forall z \in \mathcal{V}_\lambda \qquad \forall \mu \in [0, \mu_1(\lambda)] \qquad \mathrm{e}^{\theta_0} \notin \sigma(T_{\theta_0}(\lambda, x, g)W_{\theta_0}(\lambda, \mu, y, g)).$$

Indeed, the condition on C in hypothesis 2 of proposition 3.1 implies that, with $C_1 = 300$, $2^{5/2}C(1 + C^2)^2C_1 < \frac{1}{6}\sin\phi_0$ (remember $\Im\theta_0 = \phi_0$). Then (23*b*), which is satisfied with that value of C_1, implies that $\forall x$ such that $|x - 1| < \lambda^2/2$,

$$2^{5/2}\lambda \mathrm{e}^{\Re\theta_0}\eta_{\theta_0}(g)\|T_{\theta_0}(\lambda, x, g)\| < \tfrac{1}{6}\Im(\mathrm{e}^{\theta_0}). \tag{28}$$

Then from (27) and $\|g_\theta^{(r)}\|^2 < 1 + C^2$, it can be concluded that, if

$$\mathcal{V}_\lambda := \{z; |\Re z - 1| < \lambda^2/2; |\Im z| < \tfrac{1}{6}\sin\phi_0[C_1(1 + C^2)^2]^{-1}\lambda^2\}$$

and

$$\mu_1(\lambda) := (3C_2)^{-1}(1 + C^2)^{-\frac{3}{2}}\lambda^2 \cdot \|H_{\mathrm{rad}}g_{\theta_0}^{(r)}\|^{-1}\Im(\mathrm{e}^{\theta_0})$$

then

$$\forall z \in \mathcal{V}_\lambda \qquad \forall \mu \in [0, \mu_1(\lambda)] \qquad \|C_{\theta_0}(\lambda, \mu, x, y, g)\| < b_g(\lambda, \mu, y, \theta_0) < \tfrac{1}{2}\Im(\mathrm{e}^{\theta_0}) \tag{29}$$

and thus $\mathrm{e}^{\theta_0} - T_{\theta_0}(\lambda, x, g)W_{\theta_0}(\lambda, \mu, y, g)$ is invertible. As we said, this inverse is denoted by $L'_{\theta_0}(\lambda, \mu, z, g)$ and it remains to compare it with $L_{\theta_0}(\lambda, \mu, z, g)$.

3.1.3.3. $L'_{\theta_0}(\lambda, \mu, z, g)$ is an analytic continuation of $L_{\theta_0}(\lambda, \mu, z, g)$. It can be seen with formulae (11), (14), and (16)–(18) that $L'(\lambda, \mu, z, \theta_0, g)$ defined by (19) and $L(\lambda, \mu, z, \theta_0, g)$ defined by (11) coincide at points z where they are both defined. From $\eta_{\theta_0}(g) < C\lambda$ and section 3.1.2, $L(\lambda, \mu, z, \theta_0, g)$ exists for $\Im z > 2^{3/2}C\lambda^2$. As this domain has a non-empty intersection with $\mathcal{V}_\lambda$, $L'(\lambda, \mu, ., \theta_0, g)$, which is analytic in $\mathcal{V}_\lambda$, is an analytic continuation of $L(\lambda, \mu, ., \theta_0, g)$. Thus proposition 3.1 is proved. □

Of course, the analyticity of $f_2(\lambda, \mu, \ldots, g)$ results from that of $R_{2,2}(\lambda, \mu, \ldots, g)$. We will deduce the zeros of $z \to f_2(\lambda, \mu, z, g)$ near 1 and for small μ from the knowledge of the zeros of $z \to f_2(\lambda, 0, z, g)$ near 1, if we show the continuity property with respect to μ, uniformly with respect to z.

3.2. Continuity of $R_{2,2}(\lambda, \mu, z, g)$ at $\mu = 0$

Proposition 3.2. Using the same hypotheses as in proposition 3.1, $R_{2,2}(\lambda, ., z, g)$ is right-continuous at 0, uniformly for z in $\mathcal{V}_\lambda$.

Proof. One has

$$\begin{aligned}\langle 2|R(\lambda, \mu, z, g) - R(\lambda, 0, z, g)|2\rangle &= (2\lambda^2\mu)\langle 1, g_{\theta_0}^{(r)}|L(\lambda, \mu, z, \theta_0, g)\\ &\quad \times(1 \otimes H_{\mathrm{rad}})Q_2L(\lambda, 0, z, \theta_0, g)|1, g_{\theta_0}^{(r)}\rangle\\ &= (2\lambda^2\mu)\langle 1, g_{\theta_0}^{(r)}|L(\lambda, \mu, z, \theta_0, g)Q_2L(\lambda, 0, z, \theta_0, g)(1 \otimes H_{\mathrm{rad}})|1, g_{\theta_0}^{(r)}\rangle\\ &\quad +(2\lambda^2\mu)\langle 1, g_{\theta_0}^{(r)}|L(\lambda, \mu, z, \theta_0, g)[(1 \otimes H_{\mathrm{rad}}), Q_2L(\lambda, 0, z, \theta_0, g)]|1, g_{\theta_0}^{(r)}\rangle.\end{aligned}$$

By using the relations: $[(1 \otimes H_{\mathrm{rad}}), (1-X)^{-1}] = (1-X)^{-1}[(1 \otimes H_{\mathrm{rad}}), X](1-X)^{-1}$ and $[(1 \otimes H_{\mathrm{rad}}), V_{2,\theta_0}(g)] < C_0 \| H_{\mathrm{rad}} g_{\theta_0} \|$ for some C_0, one obtains, for some $C_3' \in \mathbb{R}^+$

$$\|[1 \otimes H_{\mathrm{rad}}, Q_2 L(\lambda, 0, z, \theta_0, g)]\| < C_3' \lambda \| [H_{\mathrm{rad}} g_{\theta_0} \| \| L(\lambda, 0, z, \theta_0, g)\|^2.$$

From theorem V.3.2 of [13],

$$\|L(\lambda, \mu, z, \theta_0, g)\| < \frac{|e^{\theta_0}|}{\mathrm{dist}(e^{\theta_0}, \sigma(T_{\theta_0} W_{\theta_0}))} \|T_{\theta_0}(\lambda, x, g)\|$$

consequently $\|L(\lambda, \mu, z, \theta_0, g)\| < 2C_1(1+C^2)^2 \lambda^{-2} (\sin\phi_0)^{-1}$. It then follows that there exists a number C_3 such that $\forall z \in \mathcal{V}_\lambda, \forall \mu \in [0, \mu_1(\lambda)]$

$$|\langle 2|R(\lambda, \mu, z) - R(\lambda, 0, z)|2\rangle| < C_3 \mu \lambda^{-3} \|H_{\mathrm{rad}} g_{\theta_0}\|$$

which proves the uniform continuity. □

3.3. The pole of $G_2(\lambda, \mu, \ldots, g)$ near 1 in the second sheet

This pole is the zero of $f_2(\lambda, \mu, \ldots, g)$. As we said, we are now in a position to apply Hurwitz' theorem.

Proposition 3.3. Let $\lambda \in]0, 1]$ and g satisfy the hypotheses in proposition 3.1. There then exists a neighbourhood $\mathcal{V}_\lambda$ of 1 with the following property. For every disk $\mathcal{V}'_\lambda$ with centre 1 contained in $\mathcal{V}_\lambda$, there exists $\mu_2(\lambda, \mathcal{V}'_\lambda) > 0$ such that, for $\mu \in [0, \mu_2(\lambda, \mathcal{V}'_\lambda)]$, $z \to f_2(\lambda, \mu, z, g)$ has exactly one zero in $\mathcal{V}'_\lambda$.

Proof. A simple finite-dimension calculus yields

$$R_{2,2}(\lambda, 0, z, g) = 2\lambda^2 z[z(z-1) - 2\lambda^2]^{-1} \qquad \text{for } \Im z > 0$$

and

$$f_2(\lambda, 0, z, g) = q(z)^{-1}(z-1)(z(z-2) - 4\lambda^2).$$

In the disk $\mathcal{V}_{1,\lambda}$ (centre 1, radius $\lambda^2/2$), 1 is the unique zero of $f_2(\lambda, 0, z, g)$. Therefore, thanks to the analyticity of $f_2(\lambda, \mu, ., g)$ in $\mathcal{V}_\lambda$ and to the right-continuity of $f_2(\lambda, ., z, g)$ at $\mu = 0$, uniform in $\mathcal{V}_\lambda$, proposition 3.3 is a consequence of Hurwitz' theorem. □

As zeros of $f_2(\lambda, \mu, z, g)$ are poles of

$$G_2(\lambda, \mu, z, g) = \langle 2|[z - H(\lambda, \mu, g)]^{-1}|2\rangle$$

we have just shown that the above matrix element of the resolvant of $H(\lambda, \mu, g)$ exhibits exactly one pole in every sufficiently small neighbourhood of $z = 1$, provided μ is small enough and g satisfies the hypotheses of proposition 3.1. In the following section, we are going to show that $G_1(\lambda, \mu, z, g)$ also exhibits a pole, still in the second sheet, distinct from the preceding one, and still close to 1 if λ is small (and g adequately chosen).

4. Analyticity properties of $G_1(\lambda, \mu, z, g)$ near $z = 1 + d(\lambda)$ and $\mu = 0$

It can be seen that $G_1(\lambda, 0, z, g) = z[z(z-1) - \lambda^2]^{-1}$. In the disk with centre $1 + d(\lambda)$ and radius $\frac{1}{2} + d(\lambda)$, $G_1(\lambda, 0, z, g)$ has only one pole at $z = 1 + d(\lambda)$.

As in section 3, let us now turn to the z-analyticity and μ-continuity properties of $R_{1,1}(\lambda, \mu, z, g)$ near $z = 1 + d(\lambda)$ and $\mu = 0$, in order to apply Hurwitz' theorem.

4.1. Analyticity of $R_{1,1}(\lambda, \mu, z, g)$ near $z = 1 + d(\lambda)$

For $\Im z > 0$,

$$R_{1,1}(\lambda, \mu, z, g) = \lambda^2 \int_{-\infty}^{\infty} g^2(p)(z - \mu|p|)^{-1}\,\mathrm{d}p. \tag{30}$$

Let us suppose again that g is in $L^2(\mathbb{R})$ and admits analytic continuations as in proposition 3.1. Let $\mathcal{W}_\lambda$ be the disk with centre $1 + d(\lambda)$ and radius $r = (\frac{1}{2} + d(\lambda))\sin\phi_1$, with $0 < \phi_1 < \phi_0$; it is contained in the analyticity domain of g. As $0 \notin \mathcal{W}_\lambda$, $R_{1,1}(\lambda, 0, z, g)$ is analytic in $z \in \mathcal{W}_\lambda$. For $\mu > 0$ and $z \in \mathcal{W}_\lambda$, the integrand in (30) is meromorphic with respect to p for $|p|$ in $\mu^{-1}\mathcal{W}_\lambda$, with poles only at $p = \pm\mu^{-1}z$. So, by means of a contour deformation in the p-complex plane, $R_{1,1}(\lambda, \mu, ., g)$ can be analytically continued from the region $\mathcal{W}_\lambda \cap \{z; \Im z > 0\}$ across the real axis throughout $\mathcal{W}_\lambda$.

4.2. Continuity of $R_{1,1}(\lambda, \mu, z, g)$ at $\mu = 0$, near $z = 1$

Let us suppose that $\forall\phi, |\phi| \leqslant |\phi_0|$, $p \to |p|^{1/2} g(\frac{1}{2} + \mathrm{e}^{-\mathrm{i}\phi} p)$ and $p \to |p|^{1/2} g(-\frac{1}{2} - \mathrm{e}^{-\mathrm{i}\phi} p)$ are in $L^2(\mathbb{R}^+)$. Uniform continuity is obtained by transforming

$$R_{1,1}(\lambda, \mu, z, g) - R_{1,1}(\lambda, 0, z, g) = \mu \frac{\lambda^2}{z} \int_{-\infty}^{\infty} |p| g^2(p)(z - \mu|p|)^{-1}\,\mathrm{d}p$$

into the integral on the path C shown in figure 2 below, where $\phi_0 \geqslant \phi_2 > \phi_1$; we will suppose that the integration at infinity does not give any contribution, for instance by requiring $\lim_{R\to\infty}(R \sup_{|\phi|\leqslant|\phi_0|} |g(\pm R\mathrm{e}^{\mathrm{i}\phi})|) = 0$. If $p \in C$, $z \in \mathcal{W}_\lambda$ and $\mu \in [0, 1]$, then $|(z - \mu|p|)^{-1}| < \epsilon^{-1}$, for an $\epsilon > 0$. This implies the continuity of $R_{1,1}(\lambda, ., z, g)$ at $\mu = 0$, uniformly for z in $\mathcal{W}_\lambda$.

Now Hurwitz' theorem may be applied.

4.3. The pole in the neighbourhood of 1, in the second sheet, of $G_1(\lambda, \mu, z, g)$

Proposition 4.3. Let g be as in proposition 3.1 and satisfy:

(1) $\forall\phi, |\phi| \leqslant |\phi_0|$, $p \to |p|^{1/2} g(\frac{1}{2} + \mathrm{e}^{-\mathrm{i}\phi} p)$ and $p \to |p|^{1/2} g(-\frac{1}{2} - \mathrm{e}^{-\mathrm{i}\phi} p)$ are in $L^2(\mathbb{R}^+)$

(2) $\lim_{R\to\infty}(R \sup_{|\phi|\leqslant|\phi_0|} |g(\pm R\mathrm{e}^{\mathrm{i}\phi})|) = 0$.

There then exists a neighbourhood $\mathcal{W}_\lambda$ of $1 + d(\lambda)$ with the following property.

For every disk $\mathcal{W}'_\lambda$ centred at $1 + d(\lambda)$ and contained in $\mathcal{W}_\lambda$, there exists $\mu_3(\lambda, \mathcal{W}'_\lambda)$ such that $G_1(\lambda, \mu, ., g)$ has a unique pole in $\mathcal{W}'_\lambda$ if $\mu \in [0, \mu_3(\lambda, \mathcal{W}'_\lambda)]$.

Proof. Proposition 4.3 follows from the analyticity and continuity properties just proved, due to the fact that $f_1(\lambda, 0, z, g)$ has a unique zero in $\mathcal{W}_\lambda$. □

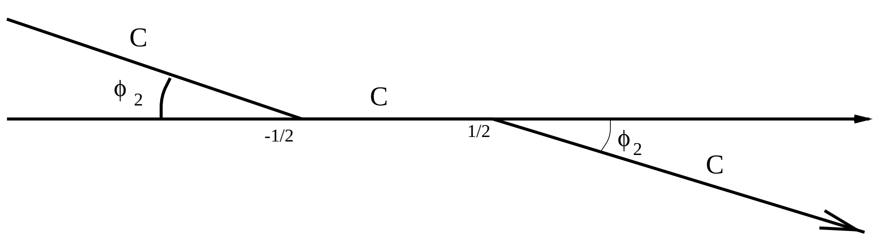

Figure 2.

5. Main proposition

Let λ be in $]0, 1]$ and g satisfy the hypotheses of propositions 3.1 and 4.3. There then exists $\mu_0(\lambda)$ such that for $\mu \in [0, \mu_0(\lambda)]$, the matrix elements of the resolvant of $H(\lambda, \mu, g)$ have at least two distinct poles in the disk $D(1, 2\lambda^2)$ with centre 1 and radius $2\lambda^2$.

Proof. In sections 3 and 4, neighbourhoods $\mathcal{V}'_\lambda$ of 1 and $\mathcal{W}'_\lambda$ of $1 + d(\lambda)$ have been considered; let us choose them to be disjoint and contained in $D(1, 2\lambda^2)$. Then, for $\mu_0(\lambda) = \min\{\mu_2(\lambda, \mathcal{V}'_\lambda), \mu_3(\lambda, \mathcal{W}'_\lambda)\}$, the two poles obtained in sections 3 and 4 respectively are distinct. □

For λ small, they are close to 1, since $d(\lambda) \sim \lambda^2$.

The result is illustrated in figure 1. Other poles have also been drawn, with their μ-dependence: they are those that would likely be obtained by also considering $E_3, E_4, \ldots,$ eigenspaces associated with eigenvalues $3, 4, \ldots$ of the operator N_{tot}.

6. Conclusion

In the simplest Hamiltonian coupling an harmonic oscillator to massless scalar bosons, we introduced an extra parameter μ, such that the 'physical' Hamiltonian is recovered for $\mu = 1$. The coupling constant λ is taken in $]0, 1]$. For μ small enough, we found two distinct poles near 1 in matrix elements of the resolvant, and they may reasonably be associated with only one level of the harmonic oscillator, the first excited one. In essence, the result may be stated in the following way. Let us consider $\mu 1 \otimes H_{\mathrm{rad}} + \lambda V(g)$ as a perturbation of $H(0, 0, g) = a^* a \otimes 1$. The eigenvalues of the unperturbed Hamiltonian $H(0, 0, g)$ are infinitely degenerated and the perturbation splits this degeneracy. We have already indicated that this point of view leads one to expect an infinity of poles for each level. If the values of the parameters in the Hamiltonian were physical, and if λ is small, it seems to us that this would be in discrepancy with the usual physical description of excited states, which affects a complex number to each level (energy and width). However, it is not strictly in contradiction with the fact proved in [2, 3] that there exists a particular complex number attached to each atomic level. The results of [2, 3] do not exclude poles different from the one the authors call the resonance. Moreover, the existence of such a number is not strictly equivalent to the fact that an excited state could be represented by a complex number. For example could not the energy of an excited state depend on the number of emitted photons?

In our study, the parameter μ cannot reach the physical value 1, far from it. However, the question is raised: how does our result extend, when μ increases to 1 and λ is small?

Appendix. The spectrum of $H(\lambda, 0, g)$

Proposition. The spectrum of the Hamiltonian $H(\lambda, 0, g)$ consists of the real numbers of the form $s_+(1 + d(\lambda)) - s_- d(\lambda)$, where s_+ and s_- are non-negative integers.

Proof. With $\omega \in]0, \pi/2]$ such that $\tan 2\omega = 2\lambda$, the transformation

$$\beta_+ = \cos\omega a \otimes 1 + \sin\omega 1 \otimes c(g)$$
$$\beta_- = -\sin\omega a \otimes 1 + \cos\omega 1 \otimes c(g)$$

leads to the following form for the Hamiltonian:

$$H(\lambda, 0, g) = (1 + d(\lambda))\beta_+^*\beta_+ - d(\lambda)\beta_-^*\beta_-.$$

The βs satisfy

$$[\beta_+, \beta_+^*] = [\beta_-, \beta_-^*] = 1 \qquad [\beta_-, \beta_+] = 0$$

and $H(\lambda, 0, g)$ is thus the difference between the Hamiltonians of two uncoupled harmonic oscillators; their energies are respectively $1 + d(\lambda)$ and $d(\lambda)$. □

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